Monotonic and Periodic Character of solutions of the Rational Difference Equation $x_{n+1} = \frac{A_n X_{n-1}}{1+X_n+X_{n-1}}$

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Abstract

We investigate the monotonic and periodic character of the nonnegative solutions of the rational difference equation

$$x_{n+1} = x_{n+1} = \frac{A_n X_{n-1}}{1 + X_n + X_{n-1}},\ n = 0, 1, \ldots,$$

where $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive real numbers.

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1 Introduction.

First we consider the autonomous difference equation:

$$X_{n+1} = \frac{AX_{n-1}}{1 + X_n + X_{n-1}} \quad , \quad n = 0, 1, 2, \ldots, \tag{1}$$

where A > 0 and the initial conditions, X_{-1} and X_0, are non-negative real numbers. The following properties were proved about Eq.(1) in [9]:

1. If A ≤ 1, then every positive solution converges to zero.
2. If A > 1, then either Eq.(1) has solutions with minimal period 2 or every positive solution of Eq.(1) converges to a period 2 cycle.

It is our goal of this paper to study the long-term behavior of the positive solutions of the non-autonomous difference equation

$$X_{n+1} = \frac{A_n X_{n-1}}{1 + X_n + X_{n-1}}, \qquad n = 0, 1, 2, \ldots, \tag{2}$$

where $\{A_n\}_{n=0}^{\infty}$ is a periodic sequence of positive real numbers with an even period and the initial conditions, X_{-1} and X_0, are non-negative real numbers. In particular, it is our goal to discover how the period(s) and the rearrangement of terms of the sequence $\{A_n\}_{n=0}^{\infty}$ affect the periodic and monotonic behavior of the solutions. In addition, it is our goal to discover the differences in the behavior of the solutions of Eq.(2) compared to the behavior of solutions of Eq.(1).

The following Theorem will show that every positive solution of Eq.(2) is bounded.

Theorem 1.1 *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2) and let $\{A_n\}_{n=0}^{\infty}$ be a sequence of positive real numbers with a finite period $k = 2, 3, 4, \ldots$ and let*

$$M = max\{A_0, A_1, \ldots, A_{k-1}\}.$$

Then for all $n \geq 1$ $X_n \leq M$.
Proof : *Observe that for all $n \geq 0$,*

$$A_n \leq M.$$

Now notice that for all $n \geq 0$,

$$X_{n+1} = \frac{A_n X_{n-1}}{1 + X_n + X_{n-1}} \leq \frac{A_n X_{n-1}}{X_{n-1}} = A_n \leq M.$$

Hence the result follows.

□

The Following Theorem by E. Camouzis and G. Ladas will be used to prove convergence of solutions of Eq.(2) to periodic cycles throughout this paper.

Theorem 1.2 *Let I be a set of real numbers, and let*

$$F : I \times I \to I$$

be a function $F(u, v)$, which decreases in u and increases in v. Then for every solution $\{X_n\}_{n=-1}^{\infty}$ of the equation

$$X_{n+1} = F(X_n, X_{n-1}); \quad n = 0, 1, 2, \ldots$$

the subsequences $\{X_{2n}\}_{n=0}^{\infty}$ and $\{X_{2n+1}\}_{n=-1}^{\infty}$ of even and odd terms are eventually monotonic.

It is interesting to note that Theorem(1.2) has a straightforward extension to non-autonomous difference equations.

2 The Case $\{A_n\}_{n=0}^{\infty}$ is periodic with period 2.

In this section we will assume that $\{A_n\}_{n=0}^{\infty}$ is periodic with minimal period 2. Now let

$$\mathrm{M} = \max\{A_0, A_1\}.$$

It is our goal to prove the following properties of Eq.(2).

1. If M $\leq$1, then every positive solution of Eq.(2) converges to zero.
2. If M $>$ 1, then Eq.(2) has period 2 solutions or every positive solution of Eq.(2) converges to the period 2 cycle.

2.1 Convergence to Zero.

In this section we will assume that M $\leq$ 1 and show that every positive solution of Eq.(2) will converge to zero. First we will prove very useful lemmas.

Lemma 2.1 *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). Suppose that $M < 1$. Then*

$$\lim_{n\to\infty} X_n = 0$$

Proof : *Notice that by iteration and inequalities, we get*

$$X_1 = \frac{A_0 X_{-1}}{1+X_0+X_{-1}} < A_0 X_{-1},\ X_3 = \frac{A_0 X_1}{1+X_2+X_1} < A_0 X_1 < A_0^2 X_{-1},\ X_5 = \frac{A_0 X_3}{1+X_4+X_3} < A_0 X_3 < A_0^3 X_{-1}, \ldots$$

Then it follows by induction that for all n $\geq$0,

$$X_{2n+1} < A_0^{n+1} X_{-1} \text{ and we see that } \lim_{n\to\infty} X_{2n+1} = 0. \tag{3}$$

Similarly, we show that for all n $\geq$1,

$$X_{2n} < A_1^n X_0 \text{ and thus } \lim_{n\to\infty} X_{2n} = 0\ . \tag{4}$$

Hence the result follows via (3) and (4). □

Lemma 2.2 *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). Suppose that $M = 1$. Then*

$$\lim_{n\to\infty} X_n = 0$$

Proof : *First we will consider the case where $A_0 = 1$ and $A_1 <1$. The case where $A_1=1$ and $A_0 <1$ is similar and will be omitted. Similarly as in Lemma(2.1), it follows by computation, iterations, and induction that for all n $\geq$ 1,*

$$X_{2n} < A_1^n X_0 \quad \text{and hence} \quad \lim_{n\to\infty} X_{2n} = 0.$$

Also by computation, iterations, and induction it follows that for all n $\geq$ 0,

$$X_{2n+1} < X_{2n-1} < \ldots < X_3 < X_1 < X_{-1}.$$

Then there exists $L_O \geq 0$ such that

$$\lim_{n\to\infty} X_{2n+1} = L_O.$$

It suffices to show that $L_O = 0$. Notice that using the properties of limits and iterations we get:

$$\lim_{n\to\infty} X_{2n+1} = \lim_{n\to\infty} \frac{A_{2n}X_{2n-1}}{1+X_{2n}+X_{2n-1}} = \frac{A_0 L_O}{1+0+L_O} = L_0.$$

Hence we see that

$$L_O = A_0 - 1 = 0.$$

□

Theorem 2.3 *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). Suppose that $M \leq 1$. Then*

$$\lim_{n\to\infty} X_n = 0.$$

Proof : *The proof follows from Lemma(2.1) and Lemma(2.2).* □

2.2 Existence of Solutions with minimal period 2.

In this section we will assume that M > 1 and show the existence of two unique solutions with minimal period 2.

Lemma 2.3 *Eq.(2) has a solution with minimal period 2 if and only if $M > 1$.*

Proof : *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). Via Theorem(2.1) we showed that when $M \leq 1$ then $\lim_{n\to\infty} X_n = 0$. Thus it suffices to consider the case where $M > 1$. Therefore we will assume that $X_{-1} \neq X_0$, and we set $X_{-1} = X_1$ and $X_2 = X_0$. By substitutions and iterations we get*

$$X_1 = \tfrac{A_0X_{-1}}{1+X_0+X_{-1}} = X_{-1},\ X_2 = \tfrac{A_1X_0}{1+X_1+X_0} = \tfrac{A_1X_0}{1+X_{-1}+X_0} = X_0, \ldots,$$

which gives us

$$A_0 = 1 + X_0 + X_{-1} \quad or \quad A_1 = 1 + X_0 + X_{-1}. \tag{5}$$

From (5) we get one of the following conditions

$$X_0 = (A_0 - 1) - X_{-1} \ or \ X_{-1} = (A_1 - 1) - X_0.$$

Now we consider the following two cases:

Case 1 : First suppose that $M = A_0 > 1$. Then we get

$$X_0 = (A_0 - 1) - X_{-1}.$$

Now observe that by iteration, it follows that

$$X_1 = \frac{A_0X_{-1}}{1+X_0+X_{-1}} = \frac{A_0X_{-1}}{1+(A_0-1-X_{-1})+X_{-1}} = \frac{A_0X_{-1}}{A_0} = X_{-1},$$

$$X_2 = \frac{A_1X_0}{1+X_1+X_0} = \frac{A_1X_0}{1+X_{-1}+X_0} = \frac{A_1X_0}{1+X_{-1}+(A_0-1-X_{-1})} = \frac{A_1X_0}{A_0} = X_0.$$

Note that $X_2 = X_0$ provided that $X_0 = 0$ as $A_1 \neq A_0$. Hence when $A_0 > 1$, then the unique period 2 cycle is

$$X_0 = 0 \ and \ X_{-1} = A_0 - 1.$$

Case 2 : Now suppose that $M = A_1 > 1$. Then we have

$$X_{-1} = (A_1 - 1) - X_0.$$

Similarly as in Case(1), we get the following unique period 2 cycle

$$X_{-1} = 0 \text{ and } X_0 = A_1 - 1. \qquad \square$$

Theorem 2.4 *Suppose that $X_{-1} > 0$, $X_0 > 0$, and $M > 1$. Then every solution of Eq.(2) converges to a period 2 cycle.*

Proof : *First recall that for all $n \geq 1$,*

$$0 < X_n < M.$$

Now let

$$F(u, v) = \frac{Av}{1 + u + v}.$$

Then we see for $u, v > 0$ that $f_u(u, v) < 0$ and $f_v(u, v) > 0$.

Now let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). Then via Theorem(1.1), $\{X_n\}_{n=-1}^{\infty}$ of Eq.(2) will have finitely many eventually monotonic subsequences. Recall that when $M > 1$, then Eq.(2) has two unique period 2 cycles. Therefore the result follows that the solution $\{X_n\}_{n=-1}^{\infty}$ of Eq.(2) will have two eventually monotonic subsequences $\{X_{2n}\}_{n=0}^{\infty}$ and $\{X_{2n+1}\}_{n=-1}^{\infty}$. $\square$

Remark 1 *From Theorem(4.10) and lemma(2.3), we can show that*

(i) If M $= A_0 > 1$, then

$$\lim_{n\to\infty} X_{2n} = 0 \text{ and } \lim_{n\to\infty} X_{2n+1} = A_0 - 1.$$

(ii) If M $= A_1 > 1$, then

$$\lim_{n\to\infty} X_{2n+1} = 0 \text{ and } \lim_{n\to\infty} X_{2n+2} = A_1 - 1.$$

3 The Case $\{A_n\}_{n=0}^{\infty}$ is periodic with period 4.

In this section we will assume that $\{A_n\}_{n=0}^{\infty}$ is periodic with minimal period 4. Now let

$$\text{M} = \max\{A_0, A_1, A_2, A_3\}, \quad P_{02} = A_0A_2 \text{ and } P_{13} = A_1A_3.$$

3.1 Convergence to Zero when $M \leq 1$.

In this section we will assume that $M \leq 1$. We will show that every positive solution of Eq.(2) will converge to zero.

Lemma 3.4 *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). Suppose that $M \leq 1$. Then*

$$\lim_{n\to\infty} X_n = 0.$$

Proof : *The case when $M < 1$ is similar to the proof in Lemma(2.1) and will be omitted. Thus we will consider the case when $M = 1$; in particular, when $A_0 = 1$, $A_1 < 1$, $A_2 = 1$, and $A_3 = 1$. All other cases are similar and will be omitted. As in lemma(2.1), by computations and inequalities, it follows by induction that for all $n \geq 1$,*

$$X_{4n} < A_1^n X_0 \text{ and } X_{4n+2} < A_1^{n+1} X_0,$$

from which we get

$$\lim_{n\to\infty} X_{4n} = 0 \text{ and } \lim_{n\to\infty} X_{4n+2} = 0.$$

Similarly we show that it follows by induction that for all $n \geq 0$,

$$X_{-1} > X_1 > \ldots > X_{2n-1} > X_{2n+1}.$$

Then there exists $L_2 \geq 0$ such that

$$\lim_{n\to\infty} X_{2n+1} = L_2.$$

It now suffices to show that $L_2 = 0$. Notice that using the properties of limits of Eq.(2) we get:

$$\lim_{n\to\infty} X_{2n+1} = \lim_{n\to\infty} \frac{A_0 X_{2n-1}}{1 + X_{2n} + X_{2n-1}} = \frac{A_0 L_2}{1 + 0 + L_2} = L_2.$$

Hence we see that

$$L_2 = A_0 - 1 = 0,$$

from which the result follows. □

3.2 Convergence to Zero when $A_0A_2 \leq 1$ and $A_1A_3 \leq 1$.

In this section we will assume that $M > 1,\ A_0A_2 \leq 1,$ and $A_1A_3 \leq 1$. We will show that every positive solution of Eq.(2) will converge to zero.

Lemma 3.5 *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). Suppose that $A_0A_2 \leq 1$ and $A_1A_3 \leq 1$. Then*

$$\lim_{n\to\infty} X_n = 0.$$

Proof : *Observe by iteration and inequalities, we get:*

$$X_1 = \frac{A_0 X_{-1}}{1 + X_0 + X_{-1}}, \quad X_3 = \frac{A_2 X_1}{1 + X_2 + X_1} < A_2 X_1 < (A_2 A_0) X_{-1} \leq X_{-1},$$

$$X_5 = \frac{A_0 X_3}{1 + X_4 + X_3} < A_0 X_3 < (A_2 A_0) X_1 \leq X_1, \quad X_7 = \frac{A_2 X_5}{1 + X_6 + X_5} < A_2 X_5 < (A_2 A_0) X_3 \leq X_3, \ldots.$$

So we see that for all $n \geq 0$,

$$X_{4n+3} < X_{4n-1} < \ldots X_7 < X_3 < X_{-1} \quad \text{and} \quad X_{4n+5} < X_{4n+1} < \ldots < X_9 < X_5 < X_1.$$

So there exists $L_1 \geq 0$ and $L_3 \geq 0$ such that

$$\lim_{n\to\infty} X_{4n+1} = L_1 \text{ and } \lim_{n\to\infty} X_{4n+3} = L_3.$$

Similarly as $A_3A_1 \leq 1$ we see that

$$X_{4n+4} < X_{4n} < \ldots X_8 < X_4 < X_0 \quad \text{and} \quad X_{4n+6} < X_{4n+2} < \ldots < X_{10} < X_6 < X_2.$$

So there exists $L_4 \geq 0$ and $L_2 \geq 0$ such that

$$\lim_{n\to\infty} X_{4n} = L_4 \text{ and } \lim_{n\to\infty} X_{4n+2} = L_2.$$

It suffices to show that

$$L_1 = L_2 = L_3 = L_4 = 0.$$

By iterations and properties of limits we get:

$$L_1 = \frac{A_0 L_3}{1 + L_4 + L_3}, \quad L_2 = \frac{A_1 L_4}{1 + L_1 + L_4}, \quad L_3 = \frac{A_2 L_1}{1 + L_2 + L_1}, \quad L_4 = \frac{A_3 L_2}{1 + L_3 + L_2}.$$

Now we will consider two cases:

Case 1 : Suppose that $A_0 A_2 = 1$. *Then notice that by iterations and inequalities*

$$L_{4n+3} = \frac{A_2 L_1}{1 + L_2 + L_1} \leq A_2 L_1 = \frac{A_2 A_0 L_3}{1 + L_4 + L_3} = \frac{L_3}{1 + L_4 + L_3}.$$

So we see that

$$L_3 \leq \frac{L_3}{1 + L_4 + L_3} \text{ and hence } \; L_4 + L_3 = 0.$$

Now observe that

$$L_1 = \frac{A_0 L_3}{1 + L_4 + L_3} = 0 \quad \text{and} \quad L_2 = \frac{A_1 L_4}{1 + L_1 + L_4} = 0.$$

Hence the result follows.

Case 2 : Suppose that $A_0 A_2 < 1$. *Note that in Case 1 we saw that*

$$X_3 < [A_0 A_2] X_{-1} \quad \text{and} \quad X_7 < [A_0 A_2]^2 X_{-1}.$$

Then it follows that

$$\lim_{n \to \infty} X_{4n+3} = 0.$$

Similarly we show that

$$\lim_{n \to \infty} X_{4n+1} = 0.$$

Furthermore, as we know that $A_1 A_3 \leq 1$, *then*

$$L_2 = \frac{A_1 L_4}{1 + L_1 + L_4} = \frac{A_1 L_4}{1 + L_4} = \frac{A_1 [\frac{A_3 L_2}{1 + L_1 + L_2}]}{1 + \frac{A_3 L_2}{1 + L_1 + L_2}} = \frac{[A_1 A_3] L_2}{1 + L_1 + L_2 + A_3 L_2} \leq \frac{[A_1 A_3] L_2}{1 + L_1 + L_2} = \frac{[A_1 A_3] L_2}{1 + L_2}.$$

If $A_1 A_3 < 1$, *then the result follows that*

$$\lim_{n \to \infty} X_{4n} = \lim_{n \to \infty} X_{4n+2} = 0.$$

If $A_1 A_3 = 1$, *then it follows that*

$$1 + L_2 = 1 \text{ and thus } \; L_2 = 0.$$

□

3.3 Existence of Solutions with Minimal Period 2.

In this section we will assume that M > 1 and show that Eq.(2) has 2 unique solutions with minimal period 2. We will assume that either

$$A_0 = A_2 > 1 \text{ or } A_1 = A_3 > 1.$$

Lemma 3.6 *Eq.(2) has a positive solution with minimal period 2 if either* $A_0 = A_2 > 1$ *or* $A_1 = A_3 > 1$.

Proof : *Let* $\{X_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(2). Via Lemma(3.4) we showed that when* $M \le 1$,

$$\lim_{n\to\infty} X_n = 0.$$

Thus it suffices to consider the case when $M > 1$. *Similarly as in Lemma(2.3) we set*

$$X_3 = X_1 = X_{-1} \quad and \quad X_4 = X_2 = X_0,$$

from which we get

$$A_0 = 1 + X_0 + X_{-1} = A_2 \quad or \quad A_1 = 1 + X_0 + X_{-1} = A_3. \tag{6}$$

From (6) and we see that either $A_0 = A_2$ *or* $A_1 = A_3$, *and we get one of the following conditions*

$$X_0 = (A_0 - 1) - X_{-1} \ or \ \ X_{-1} = (A_1 - 1) - X_0.$$

Now we consider the following two cases:

Case 1 : First suppose that $M = A_0 = A_2 > 1$*. Then we get*

$$X_0 = (A_0 - 1) - X_{-1}.$$

Now observe that by iteration, it follows that

$$X_1 = \frac{A_0 X_{-1}}{1 + X_0 + X_{-1}} = \frac{A_0 X_{-1}}{1 + (A_0 - 1 - X_{-1}) + X_{-1}} = \frac{A_0 X_{-1}}{A_0} = X_{-1},$$

$$X_2 = \frac{A_1 X_0}{1 + X_1 + X_0} = \frac{A_1 X_0}{1 + X_{-1} + X_0} = \frac{A_1 X_0}{1 + X_{-1} + (A_0 - 1 - X_{-1})} = \frac{A_1 X_0}{A_0} = X_0.$$

Note that $X_2 = X_0$ *provided that* X_0*= 0 as* $A_1 \neq A_0$*. Then we proceed with the next two iterations and we get:*

$$X_3 = \frac{A_2 X_1}{1 + X_2 + X_1} = \frac{A_0 X_{-1}}{1 + (A_0 - 1 - X_{-1}) + X_{-1}} = \frac{A_0 X_{-1}}{1 + (A_0 - 1 - X_{-1}) + X_{-1}} = X_{-1},$$

$$X_4 = \frac{A_3 X_2}{1 + X_3 + X_2} = \frac{A_3 X_0}{1 + X_{-1} + X_0} = \frac{A_3 X_0}{1 + X_{-1} + (A_0 - 1 - X_{-1})} = \frac{A_3 X_0}{A_0} = X_0.$$

Note that $A_0 = A_2$ *in order for the equalities to hold and that* $X_4 = X_2 = X_0$ *provided* $X_0 = 0$ *as* $A_3 \neq A_0$*. Hence when* $A_0 = A_2 > 1$*, then the unique period 2 cycle is*

$$X_0 = 0 \ and \ \ X_{-1} = A_0 - 1.$$

Case 2 : Now suppose that $M = A_1 = A_3 > 1$*. Then we have*

$$X_{-1} = (A_1 - 1) - X_0.$$

Similarly as in Case(1), we get the following unique period 2 cycle

$$X_{-1} = 0 \ and \ X_0 = A_1 - 1.$$

Note: This is identically the same period 2 cycle as in Theorem(2.3) of Section(2.2). □

Theorem 3.5 *Suppose that* $X_{-1} > 0$, $X_0 > 0$, *and if either* $A_0 = A_2 > 1$ *and* $A_0^2 \geq A_1A_3$ *or* $A_1 = A_3 > 1$ *and* $A_1^2 \geq A_0A_2$. *Then every solution of Eq.(2) converges to a period 2 cycle.*

Proof : *From Theorem(1.1) recall that for all* $n \geq 1$,

$$0 < X_n < M.$$

Now let

$$F(u, v) = \frac{Av}{1 + u + v}.$$

Then we see for $u, v > 0$ *that* $f_u(u, v) < 0$ *and* $f_v(u, v) > 0$.

Now let $\{X_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(2). Then via Theorem(1.1),* $\{X_n\}_{n=-1}^{\infty}$ *of Eq.(2) will have finitely many eventually monotonic subsequences. Recall that when* $M > 1$, *then Eq.(2) has two unique period 2 cycles. Therefore the result follows that the solution* $\{X_n\}_{n=-1}^{\infty}$ *of Eq.(2) will have two eventually monotonic subsequences* $\{X_{2n}\}_{n=0}^{\infty}$ *and* $\{X_{2n+1}\}_{n=-1}^{\infty}$. □

Remark 2 *From Theorem(4.10) and lemma(4.9), we can show that*

(i) $A_0 = A_2 > 1$ and $A_0^2 \geq A_1A_3$, then

$$\lim_{n\to\infty} X_{2n} = 0 \text{ and } \lim_{n\to\infty} X_{2n+1} = A_0 - 1.$$

(ii) $A_1 = A_3 > 1$ and $A_1^2 \geq A_0A_2$, then

$$\lim_{n\to\infty} X_{2n+1} = 0 \text{ and } \lim_{n\to\infty} X_{2n} = A_1 - 1.$$

3.4 Existence of Solutions with Minimal Period 4.

In this section we will show that Eq.(2) has a unique solution with minimal period 4. We will assume that

$$A_0 \neq A_2 \text{ and } A_1 \neq A_3.$$

Lemma 3.7 *Eq.(2) has a solution with minimum period 4 if either* $A_0 \neq A_2$ *and* $P_{02} > 1$ *or* $A_1 \neq A_3$, *and* $P_{13} > 1$.

Proof : *Let* $\{X_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(2) such that* $A_0 \neq A_2$ *and* $P_{02} > 1$. *The case where* $A_1 \neq A_3$ *and* $P_{13} > 1$ *is similar and will be omitted. Observe that*

$$X_1 = \frac{A_0X_{-1}}{1 + X_0 + X_{-1}} = \frac{A_0X_{-1}}{1 + X_{-1}},\ X_2 = \frac{A_1X_0}{1 + X_1 + X_0} = 0,$$

$$X_3 = \frac{A_2X_1}{1 + X_2 + X_1} = \frac{A_2X_1}{1 + X_1},\ X_4 = \frac{A_3X_2}{1 + X_3 + X_2} = \frac{A_3X_2}{1 + X_3} = 0, \ldots.$$

Now we set

$$X_3 = X_{-1}.$$

Then observe that

$$X_3 = \frac{A_2X_1}{1 + X_1} = \frac{\frac{A_2A_0X_{-1}}{1+X_{-1}}}{1 + \frac{A_0X_{-1}}{1+X_{-1}}} = \frac{A_2A_0X_{-1}}{1 + X_{-1} + A_0X_{-1}} = X_{-1}.$$

This then implies that

$$A_2A_0 = 1 + X_{-1} + A_0X_{-1} = 1 + X_{-1}(1 + A_0).$$

From the equality above, we get

$$X_{-1} = \frac{A_2A_0 - 1}{1 + A_0}.$$

Therefore proceeding with the substitutions we get

$$X_1 = \frac{A_0X_{-1}}{1 + X_{-1}} = \frac{A_0[\frac{A_2A_0-1}{1+A_0}]}{1 + [\frac{A_2A_0-1}{1+A_0}]} = \frac{A_2A_0^2 - A_0}{1 + A_0 + A_2A_0 - 1} = \frac{A_0(A_2A_0 - 1)}{A_0(1 + A_2)} = \frac{A_2A_0 - 1}{1 + A_2}.$$

Hence we see that the unique period 4 cycle is

$$X_{-1} = \frac{A_2A_0 - 1}{1 + A_0}, \quad X_0 = 0, \quad X_1 = \frac{A_2A_0 - 1}{1 + A_2}, \quad X_2 = 0.$$

Note: In this case if $A_0 = A_2$, then the period 4 cycle becomes a period 2 cycle.
Similarly, the unique period 4 cycle when $X_{-1} = 0$, $A_1 \neq A_3$, and $P_{13} > 1$, is

$$X_{-1} = 0, \quad X_0 = \frac{A_3A_1 - 1}{1 + A_1}, \quad X_1 = 0, \quad X_2 = \frac{A_3A_1 - 1}{1 + A_3}.$$

Note: In this case if $A_1 = A_3 > 1$ then the period 4 cycle becomes a period 2 cycle. □

3.5 Existence of a Positive Solution with Minimal Period 4.

In this section we will show that Eq.(2) has a unique positive solution with minimal period 4. We will assume that

$$A_0 \neq A_2, \; A_1 \neq A_3, \; P_{02} = P_{13} > 1, \; X_{-1} > 0, \text{ and } \; X_0 > 0.$$

Lemma 3.8 *Eq.(2) has a positive solution with minimum period 4 if either:*

(i) $P_{02} = P_{13} > 1, \; A_0 \neq A_2, A_1 \neq A_3, A_0 < A_2, A_1 > A_3, A_1 > A_2,$

(ii) $P_{02} = P_{13} > 1, \; A_0 \neq A_2, A_1 \neq A_3, A_0 > A_2, A_1 < A_3, A_3 > A_0,$

(iii) $P_{02} = P_{13} > 1, \; A_0 \neq A_2, A_1 \neq A_3, A_0 > A_2, A_1 > A_3, A_0 > A_1,$ *or*

(iv) $P_{02} = P_{13} > 1, \; A_0 \neq A_2, A_1 \neq A_3, A_0 < A_2, A_1 < A_3, A_2 > A_3,$

Proof : *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2) such that $P_{02} = P_{13} > 1, A_0 \neq A_2, A_1 \neq A_3, A_0 < A_2, A_1 > A_3$, and $A_1 > A_2$. The other cases are similar and will be omitted. Observe that when $X_{-1} = X_1$, we get*

$$X_1 = \frac{A_0X_{-1}}{1 + X_0 + X_{-1}} = X_{-1}.$$

This implies that

$$A_0 = 1 + X_0 + X_{-1}, \tag{7}$$

and hence from (7) and substitutions we get

$$X_2 = \frac{A_1 X_0}{1 + X_0 + X_{-1}} = \frac{A_1 X_0}{A_0}, \tag{8}$$

$$X_3 = \frac{A_2 X_1}{1 + X_2 + X_1} = \frac{A_2 X_{-1}}{1 + [\frac{A_1 X_0}{A_0}] + X_{-1}} = X_{-1}, \tag{9}$$

$$X_4 = \frac{A_3 X_2}{1 + X_3 + X_2} = \frac{A_3 X_2}{1 + X_{-1} + X_2} = \frac{A_3 [\frac{A_1 X_0}{A_0}]}{1 + X_{-1} + [\frac{A_1 X_0}{A_0}]} = X_0. \tag{10}$$

Then we see from (9) and (10) that

$$A_2 = 1 + \frac{A_1 X_0}{A_0} + X_{-1} \text{ and } \frac{A_1 A_3}{A_0} = 1 + X_{-1} + \frac{A_1 X_0}{A_0}. \tag{11}$$

Therefore from (11) we get the following relation,

$$A_0 A_2 = A_0 + A_1 X_0 + A_0 X_{-1} = A_1 A_3. \tag{12}$$

It is clear from (12) that

$$A_0 A_2 = A_1 A_3 \quad \text{and} \quad A_0 A_2 - A_0 - A_0 X_{-1} = A_0[A_2 - (1 + X_{-1})] = A_1 X_0. \tag{13}$$

Now from (7), we substitute

$$1 + X_{-1} = A_0 - X_0,$$

into (13) and we get

$$A_0 A_2 - A_0^2 + A_0 X_0 = A_1 X_0 \quad \text{and therefore} \quad A_0 A_2 - A_0^2 = X_0(A_1 - A_0).$$

Hence we see that

$$X_0 = \frac{A_0 A_2 - A_0^2}{A_1 - A_0}.$$

Therefore the unique positive period 4 cycle we get with the following pattern:

$$X_{-1} = \frac{A_0(A_1 - A_2) - A_1 + A_0}{A_1 - A_0}, \qquad X_0 = \frac{A_0 A_2 - A_0^2}{A_1 - A_0},$$

$$X_1 = \frac{A_0(A_1 - A_2) - A_1 + A_0}{A_1 - A_0}, \qquad X_2 = \frac{A_0 A_1 A_2 - A_0^2 A_1}{A_0 A_1 - A_0^2}.$$

We will also get the above unique period 4 cycle when $P_{02} = P_{13} > 1$, $A_0 \neq A_2, A_1 \neq A_3, A_0 > A_2, A_1 < A_3, A_3 > A_0$. *Similarly when* $P_{02} = P_{13} > 1$, $A_0 \neq A_2, A_1 \neq A_3, A_0 > A_2, A_1 > A_3, A_0 > A_1$, *or when* $P_{02} = P_{13} > 1$, $A_0 \neq A_2, A_1 \neq A_3, A_0 < A_2, A_1 < A_3, A_2 > A_3$, *we get another unique positive period 4 cycle with the following pattern:*

$$X_{-1} = \frac{A_1 A_2 A_3 - A_1^2 A_2}{A_1 A_2 - A_1^2}, \qquad X_0 = \frac{A_1(A_2 - A_3) - A_2 + A_1}{A_2 - A_1},$$

$$X_1 = \frac{A_1 A_3 - A_1^2}{A_2 - A_1}, \qquad X_2 = \frac{A_1(A_2 - A_3) - A_2 + A_1}{A_2 - A_1}.$$

□

Theorem 3.6 *Suppose that $X_{-1} > 0$, $X_0 > 0$, $A_0 \neq A_2$, $A_1 \neq A_3$, and either $A_0A_2 > 1$ or $A_1A_3 > 1$. Then every solution of Eq.(2) converges to a period 4 cycle.*

Proof : *From Theorem(1.1) recall that for all $n \geq 1$,*

$$0 < X_n < M.$$

Now let

$$F(u,v) = \frac{Av}{1+u+v}.$$

Then we see for $u, v > 0$ that $f_u(u,v) < 0$ and $f_v(u,v) > 0$.

Now let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). Then via Theorem(1.1), $\{X_n\}_{n=-1}^{\infty}$ of Eq.(2) will have finitely many eventually monotonic subsequences. Recall that when $P_{02} > 1$ or $P_{13} > 1$, then Eq.(2) has four unique period 4 cycles. Therefore the result follows that the solution $\{X_n\}_{n=-1}^{\infty}$ of Eq.(2) will have four eventually monotonic subsequences $\{X_{4n}\}_{n=0}^{\infty}$,$\{X_{4n+1}\}_{n=-1}^{\infty}$, $\{X_{4n+2}\}_{n=-1}^{\infty}$, and $\{X_{4n+3}\}_{n=-1}^{\infty}$. □

Remark 3 *Suppose $X_{-1} > 0$ and $X_0 > 0$. Then Eq.(2) converges to a solution with minimal period 4 if either:*

(i) $A_0 \neq A_2$, $P_{02} > 1$, and $A_0A_2 > A_1A_3$, or if $A_0A_2 = A_1A_3$, $A_0 > A_2$, $A_1 < A_3$, and $A_3 < A_0$, or if $A_0A_2 = A_1A_3$, $A_0 > A_2$, $A_1 > A_3$, and $A_0 < A_1$, then

$$\lim_{n\to\infty} X_{4n} = \lim_{n\to\infty} X_{4n+2} = 0, \lim_{n\to\infty} X_{4n-1} = \frac{A_2A_0 - 1}{1+A_0}, \text{ and } \lim_{n\to\infty} X_{4n+1} = \frac{A_2A_0 - 1}{1+A_2}.$$

(ii) $A_1 \neq A_3$, $P_{13} > 1$, and $A_1A_3 > A_0A_2$, or if $A_0A_2 = A_1A_3$, $A_0 < A_2$, $A_1 > A_3$, and $A_1 < A_2$, or if $A_0A_2 = A_1A_3$, $A_0 < A_2$, $A_1 > A_3$, and $A_1 > A_2$ then

$$\lim_{n\to\infty} X_{4n-1} = \lim_{n\to\infty} X_{4n+1} = 0, \lim_{n\to\infty} X_{4n} = \frac{A_3A_1 - 1}{1+A_1}, \text{ and } \lim_{n\to\infty} X_{4n+2} = \frac{A_3A_1 - 1}{1+A_3}.$$

(iii) If $A_0A_2 = A_1A_3$, $A_0 < A_2$, $A_1 > A_3$, and $A_1 > A_2$ or if $A_0A_2 = A_1A_3$, $A_2 < A_0$, $A_1 < A_3$, and $A_3 > A_0$, then

$$\lim_{n\to\infty} X_{4n-1} = \lim_{n\to\infty} X_{4n+1} = \frac{A_0(A_1 - A_2) - A_1 + A_0}{A_1 - A_0},$$

$$\lim_{n\to\infty} X_{4n} = \frac{A_0A_2 - A_0^2}{A_1 - A_0}, \lim_{n\to\infty} X_{4n+2} = \frac{A_0A_1A_2 - A_0^2A_1}{A_0A_1 - A_0^2}.$$

(iv) If $A_0A_2 = A_1A_3$, $A_0 > A_2$, $A_1 > A_3$, and $A_0 > A_1$ or if $A_0A_2 = A_1A_3$, $A_0 < A_2$, $A_1 < A_3$, and $A_2 > A_3$, then

$$\lim_{n\to\infty} X_{4n} = \lim_{n\to\infty} X_{4n+2} = \frac{A_1(A_2 - A_3) - A_2 + A_1}{A_2 - A_1},$$

$$\lim_{n\to\infty} X_{4n+1} = \frac{A_1A_3 - A_1^2}{A_2 - A_1}, \lim_{n\to\infty} X_{4n-1} = \frac{A_1A_2A_3 - A_1^2A_2}{A_1A_2 - A_1^2}.$$

4 The Case $\{A_n\}_{n=0}^{\infty}$ is periodic with period $2k$.

In this section we will assume that $\{A_n\}_{n=0}^{\infty}$ is periodic with minimal period 2k, such that $k = 1, 2, 3, \ldots$. Now let

$$P_1 = A_0A_2A_4 \cdots A_{2k-4}A_{2k-2}, \quad P_2 = A_1A_3A_5 \cdots A_{2k-3}A_{2k-1}, \quad \text{and } M = \max\{A_0, A_1, A_2, \ldots, A_{2k-1}\}.$$

We will investigate the monotonic and periodic nature of the solutions of Eq.(2). In particular, we will discover the existence of multiple periodic solutions of different periods of Eq.(2).

4.1 Convergence to Zero.

Theorem 4.7 *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). Suppose that $M \leq 1$. Then*

$$\lim_{n\to\infty} X_n = 0.$$

Proof : *The Proof is similar to the proof given in Lemma(2.1) and Lemma(3.4) and will be omitted.* $\square$

Theorem 4.8 *Let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). Suppose that $P_1 \leq 1$ and $P_2 \leq 1$. Then*

$$\lim_{n\to\infty} X_n = 0.$$

Proof : *The Proof is similar to the proof given in Lemma(3.5) and will be omitted.* $\square$

4.2 Existence of Solutions with Minimal Period 2.

In this section we will assume that M > 1 and show that Eq.(2) has a unique solution with minimal period 2. We will assume that either

$$A_0 = A_2 = A_4 = \cdots = A_{2k-2} > 1 \text{ or } A_1 = A_3 = A_5 = \cdots = A_{2k-1} > 1.$$

Lemma 4.9 *Eq.(2) has a positive solution with minimal period 2 if either*

(i) $A_0 = A_2 = A_4 = \cdots = A_{2k-2} > 1$ *and* $X_0 = 0$, *or*

(ii) $A_1 = A_3 = A_5 = \cdots = A_{2k-1} > 1$ *and* $X_{-1} = 0$.

Proof : *Proof follows from Theorem(2.4) and lemma(3.5) and will be omitted.* $\square$

Theorem 4.9 *Suppose that $X_{-1} > 0$, $X_0 > 0$, and if either $A_0 = A_2 = \cdots = A_{2k-2} > 1$ and $A_0^k \geq P_{135\cdots(2k-1)}$ or $A_1 = A_3 = \cdots = A_{2k-1} > 1$ and $A_1^k \geq P_{0246\cdots(2k-2)}$. Then every solution of Eq.(2) converges to a period 2 cycle.*

Proof : *From Theorem(1.1) recall that for all $n \geq 1$,*

$$0 < X_n < M.$$

Now let

$$F(u,v) = \frac{Av}{1+u+v}.$$

Then we see for $u, v > 0$ that $f_u(u,v) < 0$ and $f_v(u,v) > 0$.

Now let $\{X_n\}_{n=-1}^{\infty}$ be a positive solution of Eq.(2). Then via Theorem(1.1), $\{X_n\}_{n=-1}^{\infty}$ of Eq.(2) will have finitely many eventually monotonic subsequences. Recall that when $M > 1$, then Eq.(2) has two unique period 2 cycles. Therefore the result follows that the solution $\{X_n\}_{n=-1}^{\infty}$ of Eq.(2) will have two eventually monotonic subsequences $\{X_{2n}\}_{n=0}^{\infty}$ and $\{X_{2n+1}\}_{n=-1}^{\infty}$. $\square$

Remark 4 *From lemma(4.9) and Theorem(4.9), we can show that*

(i) $A_0 = A_2 = \cdots = A_{2k-2} > 1$ and $A_0^k \geq A_1A_3A_5\cdots A_{2k-1}$, then

$$\lim_{n\to\infty} X_{2n} = 0 \text{ and } \lim_{n\to\infty} X_{2n+1} = A_0 - 1.$$

(ii) $A_1 = A_3 = \cdots = A_{2k-1} > 1$ and $A_1^k \geq A_0A_2A_4\cdots A_{2k-2}$, then

$$\lim_{n\to\infty} X_{2n+1} = 0 \text{ and } \lim_{n\to\infty} X_{2n} = A_1 - 1.$$

4.3 Existence of Solutions with Minimal Period 4.

In this section we will show that Eq.(2) has a unique solution with minimal period 4. We will assume that 2k is a multiple of 4 and that either

$A_0 = A_4 = \cdots = A_{2k-4} \neq A_2 = A_6 \cdots = A_{2k-2}$ and $A_1 = A_5 = \cdots = A_{2k-3} \neq A_3 = A_7 = \cdots = A_{2k-1}$.

Lemma 4.10 *Eq.(2) has a solution with minimum period 4 if either:*

(i) $X_0 = 0$, $A_0 = A_4 = \cdots = A_{2k-4} \neq A_2 = A_6 \cdots = A_{2k-2}$ *, and* $P_{024\cdots(2k-2)} > 1$, *or*

(ii) $X_{-1} = 0$, $A_1 = A_5 = \cdots = A_{2k-3} \neq A_3 = A_7 = \cdots = A_{2k-1}$, *and* $P_{135\cdots(2k-1)} > 1$.

Proof : *Proof follows from lemma(3.6) and will be omitted.* □

4.4 Existence of a Positive Solution with Minimal Period 4.

In this section we will show that Eq.(2) has a unique positive solution with minimal period 4. We will assume that 2k is a multiple of 4 and that

$A_0 = A_4 = \cdots = A_{2k-4} \neq A_2 = A_6 = \cdots = A_{2k-2}$ or $A_1 = A_5 = \cdots = A_{2k-3} \neq A_3 = A_7 = \cdots = A_{2k-1}$,

$P_{024\cdots(2k-2)} = P_{135\cdots(2k-1)} > 1$, $X_{-1} > 0$, and $X_0 > 0$.

Lemma 4.11 *Eq.(2) has a positive solution with minimum period 4 if either:*

(i) $P_{02} = P_{13} > 1$, $A_0 \neq A_2, A_1 \neq A_3, A_0 < A_2, A_1 > A_3, A_1 > A_2$,

(ii) $P_{02} = P_{13} > 1$, $A_0 \neq A_2, A_1 \neq A_3, A_0 > A_2, A_1 < A_3, A_3 > A_0$,

(iii) $P_{02} = P_{13} > 1$, $A_0 \neq A_2, A_1 \neq A_3, A_0 > A_2, A_1 > A_3, A_0 > A_1$, *or*

(iv) $P_{02} = P_{13} > 1$, $A_0 \neq A_2, A_1 \neq A_3, A_0 < A_2, A_1 < A_3, A_2 > A_3$,

Proof : *Proof follows from lemma(3.7) and will be omitted.* □

Theorem 4.10 *Suppose that* $X_{-1} > 0$, $X_0 > 0$, *either* $A_0A_2 > 1$ *or* $A_1A_3 > 1$. *Then every solution of Eq.(2) converges to a period 4 cycle.*

Proof : *From Theorem(1.1) recall that for all* $n \geq 1$,

$$0 < X_n < M.$$

Now let

$$F(u,v) = \frac{Av}{1+u+v}.$$

Then we see for $u, v > 0$ *that* $f_u(u,v) < 0$ *and* $f_v(u,v) > 0$.

Now let $\{X_n\}_{n=-1}^{\infty}$ *be a positive solution of Eq.(2). Then via Theorem(1.1),* $\{X_n\}_{n=-1}^{\infty}$ *of Eq.(2) will have finitely many eventually monotonic subsequences. Recall that when* $P_{02} > 1$ *or* $P_{13} > 1$, *then Eq.(2) has four unique period 4 cycles. Therefore the result follows that the solution* $\{X_n\}_{n=-1}^{\infty}$ *of Eq.(2) will have four eventually monotonic subsequences* $\{X_{4n}\}_{n=0}^{\infty}$,$\{X_{4n+1}\}_{n=-1}^{\infty}$, $\{X_{4n+2}\}_{n=-1}^{\infty}$, *and* $\{X_{4n+3}\}_{n=-1}^{\infty}$. □

Remark 5 *Suppose* $X_{-1} > 0$ *and* $X_0 > 0$. *Then Eq.(2) converges to a solution with minimal period 4 if (i)-(x) occurs in Remark 3.*

4.5 Existence of Solutions with Minimal Period $2l;\ l \leq k$.

In this section we will assume that either:

$$P_1 > 1 \text{ and } P_1 > P_2 \quad \text{or} \quad P_2 > 1 \text{ and } P_2 > P_1,$$

and show that Eq.(2) has a unique solution with minimal period 2l.

Lemma 4.12 *Eq.(2) has a solution with minimal period 2l if*

(i) *2l divides 2k and either*

(ii) $P_1 > 1$ *and* $P_1 > P_2$ *or* $P_2 > 1$ *and* $P_2 > P_1$.

Proof : *We will assume that 2l divides 2k and consider the case where* $X_0 = 0$, $P_1 > 1$, *and* $P_1 > P_2$. *The case where* $X_{-1} = 0$, $P_2 > 1$, *and* $P_2 > P_1$ *is similar and will be omitted.*

Suppose that $X_0 = 0$, *then by iterations and substitutions we get:*

$$X_1 = \frac{A_0 X_{-1}}{1 + X_0 + X_{-1}} = \frac{A_0 X_{-1}}{1 + X_{-1}} \quad X_2 = \frac{A_1 X_0}{1 + X_1 + X_0} = 0, \quad X_3 = \frac{A_2 X_1}{1 + X_2 + X_1} = \frac{A_2 X_1}{1 + X_1},$$

$$X_4 = \frac{A_3 X_2}{1 + X_3 + X_2} = 0, \quad X_5 = \frac{A_4 X_3}{1 + X_4 + X_3} = \frac{A_4 X_3}{1 + X_3},$$

$$\vdots$$

$$X_{2l-3} = \frac{A_{2l-4} X_{2l-5}}{1 + X_{2l-4} + X_{2l-5}} = \frac{A_{2l-4} X_{2l-5}}{1 + X_{2l-5}}, \quad X_{2l-2} = \frac{A_{2l-3} X_{2l-4}}{1 + X_{2l-3} + X_{2l-4}} = 0,$$

$$X_{2l-1} = \frac{A_{2l-2} X_{2l-3}}{1 + X_{2l-2} + X_{2l-3}} = \frac{A_{2l-2} X_{2l-3}}{1 + X_{2l-3}}, \quad X_{2l} = \frac{A_{2l-1} X_{2l-2}}{1 + X_{2l-1} + X_{2l-2}} = 0.$$

Now we set

$$X_{2l-1} = X_{-1}.$$

By substitution we get that the period 2l cycle is:

$$X_{-1} = \frac{P_1 - 1}{1 + A_0 + A_0 A_2 + A_0 A_2 A_4 + \cdots + A_0 A_2 A_4 \ldots A_{2l-8} A_{2l-6} A_{2l-4}}, \quad X_0 = 0,$$

$$X_1 = \frac{P_1 - 1}{1 + A_2 + A_2 A_4 + A_2 A_4 A_6 + \cdots + A_2 A_4 A_6 \ldots A_{2l-6} A_{2l-4} A_{2l-2}}, \ X_2 = 0,$$

$$X_3 = \frac{P_1 - 1}{1 + A_4 + A_4 A_6 + A_4 A_6 A_8 + \cdots + A_4 A_6 A_8 \ldots A_{2l-4} A_{2l-2} A_0}, \ X_4 = 0,$$

$$X_5 = \frac{P_1 - 1}{1 + A_6 + A_6 A_8 + A_6 A_8 A_{10} + \cdots + A_6 A_8 A_{10} \ldots A_{2l-2} A_0 A_2}, \ X_6 = 0,$$

$$X_7 = \frac{P_1 - 1}{1 + A_8 + A_8 A_{10} + A_8 A_{10} A_{12} + \cdots + A_8 A_{10} A_{12} \ldots A_0 A_2 A_4}, \ X_8 = 0,$$

$$\vdots$$

$$X_{2l-3} = \frac{P_1 - 1}{1 + A_{2l-2} + A_{2l-2}A_0 + A_{2l-2}A_0A_2 + \cdots + A_{2l-2}A_0A_2 \ldots A_{2l-10}A_{2l-8}A_{2l-6}}, X_{2l-2} = 0,$$

$$X_{2l-1} = \frac{P_1 - 1}{1 + A_0 + A_0A_2 + A_0A_2A_4 + \cdots + A_0A_2A_4 \ldots A_{2l-8}A_{2l-6}A_{2l-4}}.$$

□

Theorem 4.11 *Suppose $X_{-1} > 0$ and $X_0 > 0$. Then Eq.(2) converges to a solution with minimal period $2l$ if either:*

$$P_1 > 1, P_1 \geq A_1A_3A_5 \cdots A_{2k-1} \text{ or } P_2 > 1, P_2 \geq A_0A_2A_4 \cdots A_{2k-2}.$$

Conjecture 1 *It is of paramount interest to determine the existence of positive periodic cycles and to what periodic cycles the solutions of Eq.(2) will converge to for $l \geq 3$.*

Example 1 *In this example we will let $k = 12$ and thereby assume $\{A_n\}_{n=0}^{\infty}$ is periodic with minimal period $2(12) = 24$. Hence when $l = 1, 2, 3, 4, 6$ or 12, there exist periodic solutions with minimal period 2,4,6,8,12, and 24 respectively.*

$\mathbf{l = 1}$: *Solutions with Minimal Period 2: Let*

$$A_0 = A_2 = A_4 = A_6 = A_8 = A_{10} = A_{12} = A_{14} = \cdots = A_{22}.$$

The Period 2 Cycle of Eq.(2) is then

$$X_{-1} = A_0 - 1, \quad X_0 = 0, \quad X_1 = A_0 - 1, \quad \text{and } X_2 = 0.$$

$\mathbf{l = 2}$: *Solutions with Minimal Period 4: Let*

$$A_0 = A_4 = A_8 = A_{12} = \cdots = A_{20}, \text{ and } A_2 = A_6 = A_{10} = A_{14} = \cdots = A_{22}.$$

The Period 4 Cycle of Eq.(2) is then

$$X_{-1} = \frac{A_2A_0 - 1}{1 + A_0}, \quad X_0 = 0, \quad X_1 = \frac{A_2A_0 - 1}{1 + A_2}, \quad X_2 = 0, \quad X_3 = \frac{A_2A_0 - 1}{1 + A_0} = X_{-1}, \quad X_4 = 0 = X_0.$$

$\mathbf{l = 3}$: *Solutions with Minimal Period 6: Let*

$$A_0 = A_6 = A_{12} = A_{18}, \; A_2 = A_8 = A_{14} = A_{20}, \text{ and } A_4 = A_{10} = A_{16} = A_{22}.$$

The Period 6 Cycle of Eq.(2) is then

$$X_{-1} = \frac{A_4A_2A_0 - 1}{1 + A_0 + A_2A_0}, \quad X_0 = 0, \quad X_1 = \frac{A_4A_2A_0 - 1}{1 + A_2 + A_4A_2}, \quad X_2 = 0,$$

$$X_3 = \frac{A_4A_2A_0 - 1}{1 + A_4 + A_4A_0}, \quad X_4 = 0, \quad X_5 = \frac{A_4A_2A_0 - 1}{1 + A_0 + A_2A_0} = X_{-1}, \quad X_6 = 0 = X_0,$$

$$X_7 = \frac{A_4A_2A_0 - 1}{1 + A_2 + A_4A_2} = X_1, \quad X_8 = 0 = X_2, \quad X_9 = \frac{A_4A_2A_0 - 1}{1 + A_4 + A_4A_0} = X_3, \quad X_{10} = 0 = X_4.$$

$\mathbf{l=6}$: *Solutions with Minimal Period 12: Let*

$$A_0 = A_{12},\ A_2 = A_{14},\ A_4 = A_{16},\ A_6 = A_{18},\ A_8 = A_{20},\ \textit{and}\ A_{10} = A_{22}.$$

The Period 12 Cycle of Eq.(2) is then

$$X_{-1} = \frac{P_{0246810} - 1}{1 + A_0 + A_2A_0 + A_4A_2A_0 + A_6A_4A_2A_0 + A_8A_6A_4A_2A_0},\ X_0 = 0,$$

$$X_1 = \frac{P_{0246810} - 1}{1 + A_2 + A_4A_2 + A_6A_4A_2 + A_8A_6A_4A_2 + A_{10}A_8A_6A_4A_2},\ X_2 = 0,$$

$$X_3 = \frac{P_{0246810} - 1}{1 + A_4 + A_6A_4 + A_8A_6A_4 + A_{10}A_8A_6A_4 + A_{10}A_8A_6A_4A_0},\ X_4 = 0,$$

$$X_5 = \frac{P_{0246810} - 1}{1 + A_6 + A_8A_6 + A_{10}A_8A_6 + A_{10}A_8A_6A_0 + A_{10}A_8A_6A_2A_0},\ X_6 = 0,$$

$$X_7 = \frac{P_{0246810} - 1}{1 + A_8 + A_{10}A_8 + A_{10}A_8A_0 + A_{10}A_8A_2A_0 + A_{10}A_8A_4A_2A_0},\ X_8 = 0,$$

$$X_9 = \frac{P_{0246810} - 1}{1 + A_{10} + A_{10}A_0 + A_{10}A_2A_0 + A_8A_4A_2A_0 + A_{10}A_6A_4A_2A_0},\ X_{10} = 0,$$

$$X_{11} = \frac{P_{0246810} - 1}{1 + A_0 + A_2A_0 + A_4A_2A_0 + A_6A_4A_2A_0 + A_8A_6A_4A_2A_0},\ X_{12} = 0.$$

5 Conclusion and Future Work.

It is of paramount interest to continue the investigation of the monotonicity and the periodicity of the poisitive solutions of Eq.(2) when $\{A_n\}_{n=0}^{\infty}$ *is periodic with an even period and with an odd period and how the delay(s) of Eq.(2) and the period of* $\{A_n\}$ *will affect the periodic character of the solutions of Eq.(2). Furthermore, it is of importance to continue the study of the following difference equations:*

(i)

$$X_{n+1} = \frac{A_nX_{n-l}}{1 + X_n + X_{n-l}}, \quad n = 0, 1, 2, \ldots$$

where $l = 2, 3, 4, \ldots$

(ii)

$$X_{n+1} = \frac{A_nX_{n-l}}{1 + X_n + X_{n-1} + \cdots + X_{n-l}}, \quad n = 0, 1, 2, \ldots$$

where $l = 2, 3, 4, \ldots$

(iii)

$$X_{n+1} = \frac{A_nX_{n-l}}{1 + B_0X_n + B_1X_{n-1} + \cdots + B_lX_{n-l}}, \quad n = 0, 1, 2, \ldots$$

where $l = 2, 3, 4, \ldots,$ *and* $\sum_{j=0}^{l} B_j \geq 0.$

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$$x_{n+1} = \frac{\alpha + \beta x_{n-1}}{\gamma + x_n} \ ,$$

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$$y_{n+1} = \frac{p + qy_n + ry_{n-1}}{1 + y_n} \ ,$$

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